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The stability of planetary waves on a sphere

By P. G. BAINES

CSIRO, Division of Atmospheric Physics, Aspendale, Victoria 3195, Australia

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The stability of individual inviscid barotropic planetary waves and zonal flow on a sphere to small disturbances is examined by means of numerical solution of the algebraic eigenvalue problem arising from the spectral form of the governing equations. It is shown that waves with total wavenumber n (the lower index of the Legendre function P_n^m which describes the waves' meridional structure) less than 3 are stable for all amplitudes, whereas those with $n \geqslant 3$ are unstable if their amplitudes are sufficiently large. For travelling waves ($m \neq 0$) with $n = 3$ and 4 and with disturbances comprised of 30 modes, the amplitudes required for instability are approximated by those obtained from triad interactions, and are smaller than those given by Hoskins (1973). For the zonal-flow modes ($m = 0$) the critical amplitudes are smaller than those predicted by triad interactions, and are close to those obtained from Rayleigh's classical criterion.

1. Introduction

The stability of Rossby waves on a β-plane has been discussed by Lorenz (1972), Hoskins & Hollingsworth (1973) and Gill (1974) and that of Rossby waves on a sphere by Hoskins (1973). Although the result that Rossby waves may be unstable is perhaps surprising from a meteorological viewpoint, it is less remarkable when viewed in the light of wave interaction theory (as was done for the case of surface waves by Hasselmann 1967*a*). It has been suggested by Lorenz (1972) and supported by Lilly (1972, 1973) that this instability is primarily responsible for the loss of predictability observed in numerical atmospheric models, i.e. the divergence observed between the properties of two time integrations with slightly different initial conditions.

Gill (1974) has shown that plane Rossby waves of both small and large amplitude are unstable, and that for small amplitudes the unstable disturbance forms a resonant triad with the primary wave. (The latter result may also be deduced from Hasselmann's (1967*b*) criterion.) McEwan & Robinson (1975) have demonstrated experimentally and theoretically that internal gravity waves in a stratified fluid are unstable and that at marginal stability this is equivalent to resonant interaction, and suggested that this process is a controlling factor for oceanic microstructure. Their analysis is based on the Mathieu equation, and the instability is 'parametric' in character. A similar type of analysis may be seen to apply to other kinds of linear wave fields where the governing equations are essentially nonlinear, so that they sustain resonant interactions, such as inertial waves in a rotating fluid. It is shown in the appendix that the stability of Rossby waves on an infinite β-plane is governed by a third-order equation akin to Mathieu's equation, which helps to elucidate its relationship with resonant interactions (Longuet-Higgins & Gill 1967) and, by analogy with the much-studied classical Mathieu equation, to obtain an overall view of the stability characteristics.

This paper is mainly concerned with Rossby waves on a rotating sphere, sometimes known as Rossby–Haurwitz waves, which have a stream function or vorticity of the form $P_n^m(\mu)\cos m(\phi-\omega_n t)$, where $\mu = \sin$ (latitude), ϕ is the longitude and $P_n^m(\mu)$ the Legendre function with m the longitudinal wavenumber and n the total wavenumber. Compared with waves on a β-plane the stability of these waves is affected by two factors: the finite size of the sphere and the discrete nature of the spectrum. As shown below, both of these tend to inhibit instability: waves with a sufficiently low wavenumber n ($\leqslant 2$) are always stable and very few planetary waves are unstable for all amplitudes.

Hoskins (1973) has considered the stability of planetary waves to a restricted class of disturbances, namely a single planetary wave and a zonal flow. Supported by some numerical integrations, he concluded that waves with zonal wavenumbers of 5 and less are stable whilst those with zonal wavenumbers greater than 5 may be unstable. However, his numerical procedure only permits waves with zonal wavenumbers which are multiples of 4, thus eliminating many destabilizing disturbances. In fact, as shown below, all waves with wavenumber $n \geqslant 3$ are unstable if their amplitudes (expressed in terms of the angular velocity of the basic rotation) are sufficiently large. Further, the critical amplitude required for instability decreases rapidly with increasing n (rather than m), and for waves with $|m| > 1$ the disturbance does not (necessarily) contain a zonal flow.

One may approach the question of the stability of waves on a sphere in a manner similar to that used in the appendix for a β-plane, arriving at a form of Mathieu equation for an arbitrary disturbance. However this is very complex because of the spherical geometry and instead the spectral form of the barotropic vorticity equation has been used, following Platzman (1962) and Hoskins (1973). The problem for the stability of any given planetary wave, including a zonal flow, may then be expressed as an algebraic eigenvalue problem, which must be truncated to a finite number of equations for numerical solution. The relevant equations are given in §2. In §3 a criterion of Fjørtoft is used to show that waves with $n < 3$ are stable regardless of their amplitude, and the eigenvalue problems for the stability of zonal flow and planetary waves are treated in §§4 and 5 respectively. For zonal flow, the amplitudes required for instability lie only slightly above those required by Rayleigh's classical criterion, namely the vanishing of the vorticity gradient somewhere in the flow, and the unstable eigenvectors show that more energy flows to lower wavenumbers than to higher ones in every case examined ($3 \leqslant n \leqslant 9$). For planetary waves P_n^m with $m \neq 0$ and $n = 3$ or 4 and P_5^4 the critical amplitudes are generally similar to those given by triad interactions alone.

In § 6 the linear stability results are tested by numerical integrations with an inviscid spherical barotropic spectral model. The waves P_3^0 and P_5^4, each with appropriate small disturbances, are discussed in some detail; in the former case the growth of the instability does not lead to destruction of the primary wave because n is small, whereas it does in the latter case.

2. Basic equations on a sphere

The equations for a non-divergent single layer of fluid on a sphere of unit radius in non-rotating spherical polar co-ordinates may be written as

$$D\zeta/Dt = 0, \quad \zeta = \nabla^2\psi, \tag{2.1a, b}$$

$$u = \frac{\partial\psi}{\partial\theta}, \quad v = -\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi}, \tag{2.1c, d}$$

where D/Dt denotes the total derivative, ζ the vorticity and ψ the stream function, θ is the co-latitude, ϕ the (east) longitude and u and v the fluid velocities in the easterly and northerly directions respectively. The dependent variables may be expressed in spectral form by expanding in spherical harmonics, e.g.

$$\zeta = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \zeta_n^m(t)\, P_n^m(\mu)\, e^{im\phi}, \tag{2.2}$$

where $\mu = \cos\theta$ and the ζ_n^m are functions of time only, the presence of any particular component ζ_n^m also implying the presence of its complex conjugate $\overline{\zeta_n^m}$. We here define the Legendre functions $P_n^m(\mu)$ such that

$$P_n^{-m}(\mu) = P_n^m(\mu), \quad \int_{-1}^{1} P_{n_1}^m(\mu)\, P_{n_2}^m(\mu)\, d\mu = \delta_{n_1 n_2}, \tag{2.3}$$

following Bourke (1972)† and Hoskins (1973), so that

$$P_n^m(\mu) = \left[\frac{2n+1}{2}\frac{(n-m)\,!}{(n+m)\,!}\right]^{\frac{1}{2}} \frac{(-1)^{m+n}(1-\mu^2)^{\frac{1}{2}m}}{2^n n\,!} \frac{d^{m+n}}{d\mu^{m+n}}(1-\mu^2)^n. \tag{2.4}$$

These functions have $n-m$ zeros in the range $(-1;1)$, and their properties are documented in many standard references (perhaps most graphically by Jahnke & Emde 1945, p. 112). For $m \neq 0$, the associated waves have $2m$ cells around a longitude circle and $n-m+1$ cells on a pole-to-pole semicircle. Throughout this paper the symbol P_n^m will be used to denote the corresponding planetary wave.

Substituting (2.2) into (2.1a), multiplying by $P_n^m(\mu)\, e^{im\phi}$ and integrating over the whole sphere yields the spectral barotropic vorticity equation (Silberman 1954; Platzman 1962; Hoskins 1973; but with slightly different notation)

$$\frac{d\zeta_\gamma}{dt} = i \sum_{\alpha<\beta} I_{\gamma\beta\alpha}\, \zeta_\beta\, \zeta_\alpha, \tag{2.5}$$

† Bourke defines $P_n^{-m}(\mu) = (-1)^m P_n^m(\mu)$.

where γ, β and α denote pairs (m_γ, n_γ), etc., and

$$I_{\gamma\beta\alpha} = \begin{cases} 0 \quad \text{if} \quad m_\gamma \neq m_\alpha + m_\beta, \\ \left(\dfrac{1}{n_\beta(n_\beta+1)} - \dfrac{1}{n_\alpha(n_\alpha+1)}\right) K_{\gamma\beta\alpha} \quad \text{if} \quad m_\gamma = m_\alpha + m_\beta, \end{cases} \tag{2.6}$$

where
$$K_{\gamma\beta\alpha} = \int_{-1}^{1} P_\gamma \left(m_\beta P_\beta \frac{dP_\alpha}{d\mu} - m_\alpha P_\alpha \frac{dP_\beta}{d\mu} \right) d\mu. \tag{2.7}$$

$K_{\gamma\beta\alpha}$ may also be written as K with a subscript $n_\gamma n_\beta n_\alpha$ and superscript $m_\gamma m_\beta m_\alpha$. $\sum_{\alpha<\beta}$ denotes summation over all pairs (α, β), but without repetition (i.e. no permutations). The coefficients $K_{\gamma\beta\alpha}$ have been evaluated by Silberman (1954), and are zero unless

$$\left. \begin{aligned} &|n_\alpha - n_\beta| \not> n_\gamma < n_\alpha + n_\beta, \quad n_\alpha + n_\beta + n_\gamma = \text{odd}, \\ &(m_\beta, n_\beta) \neq (-m_\gamma, n_\gamma), \quad (m_\alpha, n_\alpha) \neq (-m_\gamma, n_\gamma). \end{aligned} \right\} \tag{2.8}$$

We also have the redundancy relations

$$K_{\gamma\beta\alpha} = -K_{\gamma\alpha\beta} = K_{\alpha\bar{\beta}\gamma} = K_{\beta\gamma\bar{\alpha}}, \tag{2.9}$$

where $\bar{\alpha}$ and $\bar{\beta}$ denote $(-m_\alpha, n_\alpha)$ and $(-m_\beta, n_\beta)$. These coefficients have been calculated numerically when necessary using routines developed by Dr W. Bourke of the Commonwealth Meteorological Research Centre, based on expressions given by Silberman (1954).

It may be readily shown (e.g. Platzman 1962) that the component containing $P_1^0(\mu) = (\frac{3}{2})^{\frac{1}{2}} \mu$ contains the total angular momentum about the co-ordinate axis, and is equivalent to a rotation about this axis with angular velocity Ω given by

$$\Omega = (\tfrac{3}{8})^{\frac{1}{2}} \zeta_1^0. \tag{2.10}$$

Hence (2.5) may be written as

$$\frac{d\zeta_\gamma}{dt} = -im_\gamma \left(1 - \frac{2}{n_\gamma(n_\gamma+1)}\right) \Omega \zeta_\gamma + i \sum_{\substack{\alpha<\beta \\ \alpha, \beta \neq (0,1)}} I_{\gamma\beta\alpha} \zeta_\beta \zeta_\alpha, \tag{2.11}$$

and the linearized solutions to these equations are the Rossby–Haurwitz waves. These are also exact solutions to (2.11), as indeed is any sum of components with the same degree n (Craig 1945; Neamtan 1946).

We consider the stability of a single wave ζ_{α_1} (hereafter referred to as the primary wave) to infinitesimal disturbances. If ζ_{α_1} represents a zonal flow the appropriate linearized form of (2.11) for the disturbances is

$$\frac{d\zeta_\gamma}{dt} = -im_\gamma \left(1 - \frac{2}{n_\gamma(n_\gamma+1)}\right) \Omega \zeta_\gamma + i \zeta_{\alpha_1} \sum_{\substack{\beta \\ \beta \neq (0,1)}} I_{\gamma\beta\alpha_1} \zeta_\beta. \tag{2.12}$$

If ζ_{α_1} represents a planetary wave it is best to consider the equations in a frame of reference rotating with angular velocity

$$\Omega' = \Omega[1 - 2/n_{\alpha_1}(n_{\alpha_1}+1)],$$

which is the angular velocity of the wave pattern

$$2\zeta_{\alpha_1} P_n^m(\mu)\cos m\phi \quad (m = m_{\alpha_1}, n = n_{\alpha_1}).$$

The vorticity equation in this rotating frame is

$$D\zeta/Dt + 2\Omega'\,\partial\psi/\partial\phi = 0, \tag{2.13}$$

which in spectral form becomes

$$\frac{d\zeta_\gamma}{dt} - 2\Omega'\frac{im_\gamma}{n_\gamma(n_\gamma+1)}\zeta_\gamma = -im_\gamma(\Omega-\Omega')\left(1-\frac{2}{n_\gamma(n_\gamma+1)}\right)\zeta_\gamma + i\sum_{\substack{\alpha<\beta \\ \beta,\alpha\neq(0,1)}} I_{\gamma\beta\alpha}\zeta_\beta\zeta_\alpha, \tag{2.14}$$

where the notation is the same as above except that the time dependence of the coefficients will be different. The linearized disturbance equations then are

$$\begin{aligned}\frac{d\zeta_\gamma}{dt} = &-im_\gamma 2\Omega\left(\frac{1}{n_{\alpha_1}(n_{\alpha_1}+1)} - \frac{1}{n_\gamma(n_\gamma+1)}\right)\zeta_\gamma \\ &+ i\zeta_{\alpha_1}\Sigma\left(\frac{1}{n_\beta(n_\beta+1)} - \frac{1}{n_{\alpha_1}(n_{\alpha_1}+1)}\right)K_{\gamma\beta\alpha_1}\zeta_\beta \\ &+ i\zeta_{\alpha_1}\Sigma\left(\frac{1}{n_\beta(n_\beta+1)} - \frac{1}{n_{\alpha_1}(n_{\alpha_1}+1)}\right)K_{\gamma\beta\bar{\alpha}_1}\zeta_\beta,\end{aligned} \tag{2.15}$$

where ζ_{α_1} is real in this case and the first summation is over all β with

$$m_\beta = m_\gamma - m_{\alpha_1},$$

the second over all β with $m_\beta = m_\gamma + m_{\alpha_1}$.

3. Fjørtoft's theorem and its implications for stability

Conservation of energy in the system governed by (2.1) yields

$$E = \frac{1}{2}\int_S (\nabla\psi)^2\,dS = -\frac{1}{2}\int_S \psi\nabla^2\psi\,dS, \tag{3.1}$$

integrating over the whole sphere. Since

$$\nabla^2 P_n^m(\mu)\,e^{im\phi} = -n(n+1)\,P_n^m(\mu)\,e^{im\phi},$$

we obtain

$$E = \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=-n}^{n} 2\pi\frac{\zeta_n^m\overline{\zeta_n^m}}{n(n+1)} = \pi\sum_{n=1}^{\infty}\frac{A_n^2}{n(n+1)} = \sum_{n=1}^{\infty}E_n = \text{constant}, \tag{3.2}$$

where

$$A_n^2 = \sum_{m=-n}^{n}\zeta_n^m\overline{\zeta_n^m}, \quad E_n = \pi\frac{A_n^2}{n(n+1)}. \tag{3.3}$$

A further continuous infinity of invariants may be obtained from the vorticity equation, viz.,

$$\frac{D}{Dt}\int_S \zeta^\eta\,dS = 0, \tag{3.4}$$

where η is any real or complex number with $\mathrm{Re}\,\eta \geqslant 1$. Equation (3.4) with $\eta = 1$ is trivial and equivalent to conservation of angular momentum. $\eta = 2$ yields a second quadratic invariant commonly known as the enstrophy:

$$F = \frac{1}{2}\int_S \zeta^2 dS = \pi \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \zeta_n^m \overline{\zeta_n^m}, \tag{3.5}$$

therefore

$$F = \pi \sum_{n=1}^{\infty} A_n^2 = \text{constant}. \tag{3.6}$$

From equations (equivalent to) (3.2) and (3.6) Fjørtoft (1953) deduced that any energy exchange must take place between components with (at least) three different values of n, and if one of three components represents a source or sink for both the other two, its n value must lie between those of the latter. He also deduced some inequalities concerning the spectral distribution of energy, the most important of which (Fjørtoft's equation 44) is, in the present notation,

$$\sum_{n=N+1} E_n/E < \frac{F/E - 2}{(N+1)(N+2) - 2}, \tag{3.7}$$

for specified E and F and any positive integer N. Now the component P_1^1 represents rigid rotation about an axis perpendicular to the co-ordinate axis, and hence like P_1^0, its magnitude (taken as zero here) must remain constant by conservation of angular momentum. This permits the strengthening of Fjørtoft's above inequality by using the same argument as his but omitting the modes with $n = 1$, giving

$$\sum_{n=N+1}^{\infty} E_n/E_{2+} < \frac{F_{2+}/E_{2+} - 6}{(N+4)(N-1)}, \tag{3.8}$$

where $N \geqslant 2$ and E_{2+} and F_{2+} are the energy and enstrophy omitting modes with $n = 1$. If all the energy is initially in a mode or modes with $n = L$ this may be written as

$$\sum_{n=N+1}^{\infty} E_n/E_{2+} < \frac{(L+3)(L-2)}{(N+4)(N-1)}. \tag{3.9}$$

If we consider the stability of individual modes for infinitesimal disturbances, P_1^0 and P_1^1 must be absolutely stable by conservation of angular momentum, and (3.9) shows that if all energy is initially in modes with $n = 2$ it must remain there. The interaction rules [equation (2.9)] also show that energy cannot flow between P_2^0, P_2^1 and P_2^2 directly, so that these modes must all be stable, regardless of their amplitudes. Equation (3.9) indicates that modes with $n \geqslant 3$ may be unstable, and it is shown below that this is in fact the case if their amplitudes are sufficiently large, although the amount of energy which may flow to higher wavenumbers n is limited.

4. The stability of zonal flow

We consider the stability of zonal flows whose vorticity (or stream function) is represented by a single Legendre polynomial P_n^0, where n is an integer. From the preceding section we know that P_1^0 and P_2^0 are completely stable, and here we consider values of n ranging from 3 to 9. The spectral equations governing infinitesimal disturbances are given by (2.12), an infinite set of linear equations with constant coefficients. We look for the conditions under which this system has unstable normal modes. Assuming exponential time dependence $e^{i\omega t}$ equations (2.12) may be transformed into an infinite set of algebraic equations, viz.,

$$\sum_{\beta} \left\{ I_{\gamma\beta\alpha_1} \zeta_{\alpha_1} - \delta_{\beta\gamma} \left[m_\beta \left(1 - \frac{2}{n_\beta(n_\beta+1)} \right) \Omega + \omega \right] \right\} \zeta_\beta = 0. \tag{4.1}$$

The condition for non-trivial solutions is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0, \tag{4.2}$$

where 'det' denotes 'determinant', $\lambda = \omega/\Omega$ and the matrix $\mathbf{A}$ has elements

$$a_{\beta\gamma} = \xi_{\alpha_1} I_{\gamma\beta\alpha_1} - \delta_{\beta\gamma} m_\beta \left(1 - \frac{2}{n_\beta(n_\beta+1)} \right), \tag{4.3}$$

where $\xi_{\alpha_1} = \zeta_{\alpha_1}/\Omega$. Since m_{α_1} is zero, (2.6) requires m_β and m_γ to be equal for non-zero interaction coefficients. This implies that the set of equations for ζ_β with $m_\beta = 1$ is decoupled from that with $m_\beta = 2$, and so on, so that these groups of equations may be considered separately. In other words, the equations governing disturbances of given zonal wavenumber form an independent set.

The eigenvalue problem represented by (4.2), for values of m_β ranging from 1 to $n_{\alpha_1} - 1$, was solved numerically for truncated systems with given values of ξ_{α_1}. ξ_{α_1} was increased from zero to the point where complex eigenvalues λ first appeared, yielding the critical amplitude for instability of mode α_1 with the given truncation. The truncation number N_T (i.e. the rank of the matrix $\mathbf{A}$, or the number of equations) was varied up to 20 in the important cases, and the values obtained for the critical amplitudes appeared to converge as N_T increased, with an apparent approximation to within at least two significant figures to the limiting value where $N_T = 20$. In most cases the critical amplitude decreased monotonically with increasing truncation number (by an amount which depended on the wavenumber but was typically of order $\frac{1}{2}$, overall), so that the value for $N_T = 20$ was taken as the critical amplitude.

The eigenvalues and eigenvectors were obtained by employing library subroutines provided by the CSIRO Division of Computing Research. Routine EIGENP, based on an algorithm by Grad & Brebner (1968) using Householder's method (Wilkinson 1965, p. 290), was mostly used, with routine EIGEN, based on a method due to Eberlein (1962), used as a check on certain cases, notably those in which the eigenvalues did not decrease monotonically with increasing N_T. In every case so checked the eigenvalues obtained by the two methods agreed to at least six significant figures.

The critical amplitudes of the vorticity of the zonal modes, expressed in terms of the angular velocity of the rotation about the co-ordinate axis together with the zonal wavenumber of the most unstable disturbance, are given in table 1. With the earth's rotation and radius the associated equatorial wind speeds are 89·3 ('trade winds') and 29·5 m/s for $\pm P_3^0$ respectively, 25·7 and 5·44 m/s for $\pm P_5^0$, and 9·97 and 1·8 m/s for $\pm P_7^0$. The mean barotropic zonal flow of the

Wave	Vorticity amplitude A	Zonal wavenumber M	Energy $E = A^2/N(N+1)$
$+P$	0·799	2	$5{\cdot}32 \times 10^{-2}$
$-P$	0·264	1	$5{\cdot}81 \times 10^{-3}$
P_4^0	0·161	1	$1{\cdot}30 \times 10^{-3}$
$+P_5^0$	0·362	3	$4{\cdot}90 \times 10^{-3}$
$-P_5^0$	0·0777	1	$2{\cdot}01 \times 10^{-4}$
P_6^0	0·0564	1	$7{\cdot}57 \times 10^{-5}$
$+P_7^0$	0·1957	3	$6{\cdot}84 \times 10^{-4}$
$-P_7^0$	0·0353	1	$2{\cdot}23 \times 10^{-5}$
P_8^0	0·0292	1	$1{\cdot}18 \times 10^{-5}$
$+P_9^0$	0·122	1	$1{\cdot}65 \times 10^{-4}$
$-P^0$	0·0204	1	$4{\cdot}62 \times 10^{-6}$

TABLE 1. Critical amplitudes for the vorticity and energy of the zonal-flow modes, together with the zonal wavenumber of the most unstable disturbance.

atmosphere, represented approximately by $+P_3^0$, is accordingly very stable by this criterion. The corresponding energy amplitudes are plotted in figure 1, which also shows the necessary amplitudes for instability obtained from Rayleigh's criterion for instability of a shear flow. On a sphere this has the form

$$d\zeta(\mu)/d\mu + 2\Omega = 0 \quad \text{somewhere,} \tag{4.4}$$

where $\zeta(\mu)$ is the vorticity of the zonal flow. If $\zeta(\mu) = \zeta_n^0 P_n^0(\mu)$ this becomes

$$\frac{\zeta_n^0}{\Omega}\frac{d}{d\mu}P_n^0(\mu) + 2 = 0. \tag{4.5}$$

It may be seen that in every case the actual amplitude required for instability lies only slightly above the necessary value obtained from (4.5).

The stability criteria separate the modes into two distinct groups, each with its own approximate power-law dependence on n (see figure 1): modes P_3^0, P_5^0, P_7^0, etc., with *positive* coefficients are much more stable than the others. In addition, all the modes of the second group (even modes and negative odd modes) are most unstable to disturbances of zonal wavenumber one, which is not the case for the first group.

An inspection of the eigenvectors $(\zeta_{n_1}^m, \zeta_{n_2}^m, \ldots)$ representing the most unstable disturbances to each zonal-flow harmonic P_n^0 indicates that they have no uniform structure other than a general tendency for the magnitude of the components $\zeta_{n_i}^m$ to decrease (not always monotonically) as n_i increases. If n is odd, all the components with n_i odd are zero (i.e. the most unstable disturbance has n_i even). Integral constraints on the unstable eigenvectors may be obtained as follows. The equation for an infinitesimal disturbance ψ_2 to an initial zonal flow ψ_1 with total wavenumber L in a rotating system is

$$\frac{\partial}{\partial t}\nabla^2\psi_2 + \frac{1}{\sin\theta}(\psi_{1\theta}\nabla^2\psi_{2\phi} - \psi_{2\phi}\nabla^2\psi_{1\theta}) + 2\Omega\psi_{2\phi} = 0, \tag{4.6}$$

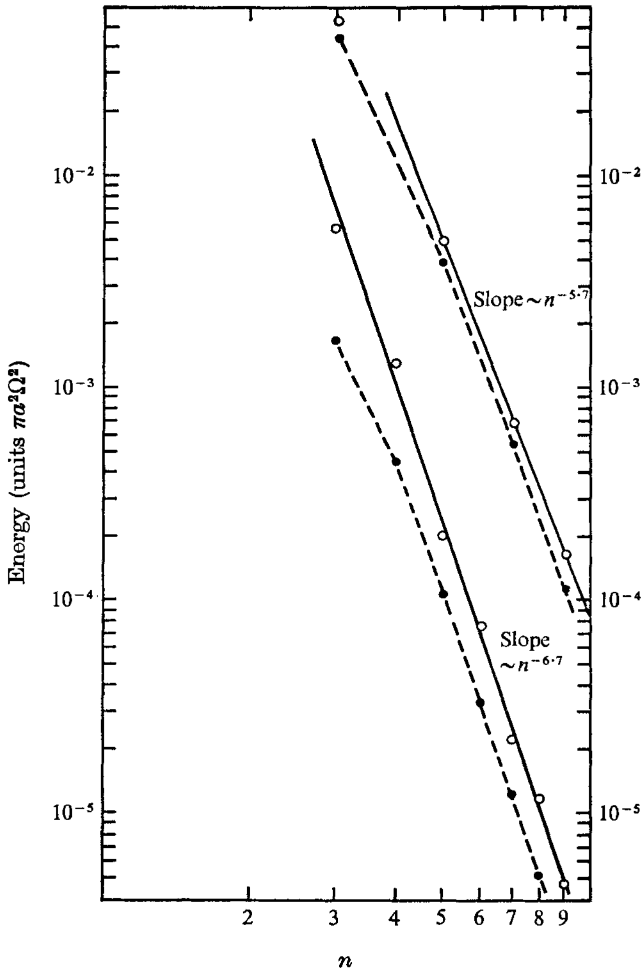

FIGURE 1. Critical amplitudes in terms of energy (units $\pi a^2\Omega^2$, where a is the earth's radius) for the zonal-flow modes P_N^0. The continuous lines drawn have the slope indicated and the dotted lines connect the corresponding points obtained from Rayleigh's criterion.

where the suffixes θ and ϕ denote derivatives. Multiplying by ψ_2, integrating over the sphere and invoking Green's theorem yields

$$\frac{\partial}{\partial t}\int_S \tfrac{1}{2}(\nabla\psi_2)^2 dS = \int_{-1}^{1}\frac{d\psi_1}{d\mu}\int_0^{2\pi}\psi_2\nabla^2\psi_{2\phi}\,d\phi\,d\mu = w, \tag{4.7}$$

where w is the rate of working by 'Reynolds stresses'. Multiplying (4.6) by $\nabla^2\overline{\psi}_2$ yields

$$\frac{\partial}{\partial t}\frac{1}{2}\int_S (\nabla^2\psi_2)^2 dS = L(L+1)\,w, \tag{4.8}$$

and eliminating w between (4.7) and (4.8) gives (Karunin 1970)

$$\partial F^1/\partial t = L(L+1)\,\partial E^1/\partial t, \tag{4.9}$$

where E^1 and F^1 are the energy and enstrophy of the disturbance, which may be expressed as

$$E^1 = \sum_{i=2}^{k} E^1_{n_i}, \quad F^1 = \sum_{n_i=2}^{k} n_i(n_i+1) E^1_{n_i}, \tag{4.10}$$

where ideally k is infinite and the sum does not include $n_i = 1$ or L since these form no part of an unstable disturbance. For a growing mode, each E_{n_i} may be written as

$$E_{n_i} = \hat{E}_{n_i} e^{\lambda t},$$

where λ has a positive real part and the $\hat{E}_{n_i}$ are constants. Substituting in (4.9) and normalizing the $\hat{E}_{n_i}$ yields

$$\sum_{n_i=2}^{k} \hat{E}_{n_i} = 1, \quad \sum_{n=2}^{k} n_i(n_i+1) \hat{E}_{n_i} = L(L+1). \tag{4.11}$$

From these relations we may deduce from Fjørtoft's argument

$$\mathscr{E}_N = \sum_{n_i=N+1}^{\infty} \hat{E}_{n_i} < \frac{(L+3)(L-2)}{(N+4)(N-1)}, \tag{4.12}$$

which corresponds exactly to (3.9) except that $\hat{E}_L$ is necessarily zero.

Equation (4.12) refers to the relative rate of flow of energy to modes with wavenumber greater than N, and if $N = L$, the upper bound on $\mathscr{E}_N$ is less than $\frac{1}{2}$ only for $L = 3$. It has been pointed out by Merilees & Warn (1975) that Fjørtoft's argument for triad interactions that more energy must flow to lower wavenumbers than to higher ones when energy is lost by a single mode is spurious, although this happens in a majority of cases. However, for the truncated eigenvectors obtained numerically with N_T varying between 12 and 20, $\mathscr{E}_L$ was always less than 0·5, implying a greater flux of energy to the lower wavenumbers. Also, the corresponding ratios of the flux of mean-square vorticity to higher wavenumbers to the flux to lower wavenumbers are close to unity for all $L < 9$ except $L = 3$, where they have values of 2·1 (positive vorticity coefficient) and 1·65 (negative vorticity coefficient).

5. The stability of planetary waves

The stability of planetary waves on a sphere has been discussed by Hoskins (1973), who considered perturbations consisting of a single planetary wave together with a zonal flow. This form of disturbance was chosen because of the results of numerical integrations, which were, however, restricted to zonal wavenumbers which are multiples of 4, so that other types of disturbance were prohibited.

We consider first of all the stability of planetary waves to perturbations consisting of (*a*) a single planetary wave and a zonal flow as per Hoskins (1973) and (*b*) two planetary waves neither of which is a zonal flow. The criteria for instability due to disturbances of type (*a*) may be obtained from equation (4.17) of Hoskins' paper (to which the reader is referred for the relevant analysis), and these are presented in table 2 for all planetary waves up to a wavenumber of

$N \backslash M$	1	2	3	4	5	6	7	8	9
3	2·0	—	2·087						
4	0·7734	—	1·551	0·8248					
5	0·3972	—	0·8680	0·6316	0·5161				
6	0·2359	—	0·5101	0·4540	0·4155	0·3662			
7	0·1535	1·1974	0·3278	0·3245	0·3218	0·3062	0·2775		
8	0·1065	0·3601	0·2258	0·2368	0·2472	0·2483	0·2390	0·2194	
9	0· 07737	0·2148	0·1638	0·1774	0·1914	0·1994	0·2001	0·1933	0·1787

TABLE 2. Critical amplitudes for the vorticity $|\zeta_N^M/\Omega|$ of planetary waves P_N^M subject to disturbances consisting only of a single planetary wave and a zonal flow, as in the analysis of Hoskins (1973).

9.† For an initial unstable wave P_n^1 the most destabilizing disturbance consists of the wave P_{n-1}^1 plus the interacting zonal-flow harmonics, which act in unison; for an initial wave P_n^m with $m \geqslant 2$ the disturbance is P_{n+1}^m plus zonal flow.

For disturbances of type (*b*) we consider (2.15) with only two disturbance components ζ_1 and ζ_2 (and their complex conjugates), with wavenumbers m_1, n_1, m_2 and n_2, where $m_1, m_2 > 0$. Consideration of the selection rules (2.6) and (2.8) shows that the only mutually interacting pairs are those which have $m_1 \pm m_2 = m_\alpha$, the zonal wavenumber of the primary wave. It transpires that, for every wave with $m_\alpha > 1$, the members of the triad with the lowest critical amplitude satisfy the condition $m_1 + m_2 = m_\alpha$, and not $m_1 - m_2 = m_\alpha$; if $m_\alpha = 1$ the critical amplitudes for all cases $m_1 - m_2 = m_\alpha$ are greater than those given in table 2. The equations for disturbances with $m_1 + m_2 = m_\alpha$ take the form

$$\frac{d\zeta_1}{dt} = -im_1 2\Omega\left(\frac{1}{n_\alpha(n_\alpha+1)} - \frac{1}{n_1(n_1+1)}\right)\zeta_1 + i\zeta_\alpha\left(\frac{1}{n_2(n_2+1)} - \frac{1}{n_\alpha(n_\alpha+1)}\right)K_\pi^\sigma \bar{\zeta}_2, \tag{5.1a}$$

$$\frac{d\bar{\zeta}_2}{dt} = im_2 2\Omega\left(\frac{1}{n_\alpha(n_\alpha+1)} - \frac{1}{n_2(n_2+1)}\right)\bar{\zeta}_2 - i\zeta_\alpha\left(\frac{1}{n_1(n_1+1)} - \frac{1}{n_\alpha(n_\alpha+1)}\right)K_\rho^\tau \zeta_1, \tag{5.1b}$$

where $\sigma = m_1 - m_2\, m_\alpha$, $\pi = n_1\, n_2\, n_\alpha$, $\tau = m_2 - m_1\, m_\alpha$ and $\rho = n_2\, n_1\, n_\alpha$, together with their complex conjugates. Eliminating $\bar{\zeta}_2$ gives

$$\frac{d^2\zeta_1}{dt^2} + ib_1\frac{d\zeta_1}{dt} + b_2\zeta_1 = 0, \tag{5.2}$$

where

$$b_1 = 2\Omega\left(\frac{m_1 - m_2}{n_\alpha(n_\alpha+1)} + \frac{m_2}{n_2(n_2+1)} - \frac{m_1}{n_1(n_1+1)}\right), \tag{5.3a}$$

$$b_2 = \left(\frac{1}{n_2(n_2+1)} - \frac{1}{n_\alpha(n_\alpha+1)}\right)\left(\frac{1}{n_1(n_1+1)} - \frac{1}{n_\alpha(n_\alpha+1)}\right)(\zeta_\alpha^2 K^2 + 4\Omega^2 m_1 m_2), \tag{5.3b}$$

$$K = K_\pi^\sigma = -K_\rho^\tau.$$

† The critical values for the waves P_8^7 and P_9^8 agree with the lowest curve of figure 5(*b*) of Hoskins (ζ_{rms} apparently representing $|\zeta_n^m|$), but the values for P_{M+1}^M, $M \leqslant 6$, do not.

$N \backslash M$	2	3	4	5	6	7	8	9
3	0·4536 P^1_2, P^1_4	0·2036 P^1_2, P^2_4						
4	0·0693 P^1_3, P^1_6	0·2102 P^1_3, P^2_6	0·4584 P^2_3, P^2_6					
5	0·031 P^1_4, P^1_6	0·1735 P^1_3, P^2_7	0·0402 P^1_3, P^3_7	0·0870 P^2_4, P^3_6				
6	0·00927 P^1_5, P^1_8	0·00768 P^1_4, P^2_9	0·0132 P^2_5, P^2_8	0·0346 P^1_4, P^4_7	0·0375 P^3_5, P^3_8			
7	0·0573 P^1_5, P^1_{11}	0·00728 P^2_6, P^1_{12}	0·0578 P^1_4, P^3_{10}	0·0 P^2_6, P^3_8	0·0521 P^2_5, P^4_9	0·0497 P^3_6, P^4_8		
8	0·0191 P^1_6, P^1_{13}	0·0201 P^2_7, P^1_{14}	0·0309 P^1_5, P^3_{12}	0·00613 P^1_5, P^4_{10}	0·023 P^4_7, P^2_{12}	0·0331 P^4_7, P^3_{10}	0·0558 P^2_6, P^6_9	
9	0·0028 P^1_7, P^1_{15}	0·0 P^1_6, P^2_{14}	0·00599 P^2_7, P^2_{15}	0·00266 P^1_5, P^4_{13}	0·0233 P^3_7, P^3_{15}	0·0120 P^1_5, P^6_{11}	0·00695 P^5_8, P^3_{12}	0·0 P^3_6, P^6_{14}

TABLE 3. Critical amplitudes for the vorticity $|\zeta^M_N/\Omega|$ of planetary waves P^M_N subject to disturbances consisting only of two planetary waves neither of which is a zonal flow. The two waves forming the most unstable disturbance for each P^M_N are indicated.

Solutions of (5.2) of the form $e^{\mu t}$ will be unstable if

$$\left(\frac{2\zeta_\alpha}{\Omega}\right)^2 \left(\frac{1}{n_1(n_1+1)} - \frac{1}{n_\alpha(n_\alpha+1)}\right)\left(\frac{1}{n_\alpha(n_\alpha+1)} - \frac{1}{n_2(n_2+1)}\right)K^2 > 4\left(\frac{m_\alpha}{n_\alpha(n_\alpha+1)} - \frac{m_1}{n_1(n_1+1)} - \frac{m_2}{n_2(n_2+1)}\right)^2, \quad (5.4)$$

which requires n_α to lie between n_1 and n_2 as expected. The minimum amplitudes $A = 2\zeta_\alpha/\Omega$ for which inequality (5.4) is satisfied for each primary wave, and the two components of the corresponding unstable triad, are given in table 3. Except for primary waves with a zonal wavenumber of one the critical amplitudes are all considerably smaller than the corresponding ones in table 2, so that the preferred form of disturbance consists of two planetary waves. For any initial wave P^M_N there are a number of unstable triads, and those given in table 3 are only those with the lowest critical amplitudes. For three waves, namely P^5_7, P^2_9 and P^9_9, the critical amplitudes are zero because of the vanishing of the last term in (5.4). This is in fact just the frequency condition for resonant interaction between these waves and the other members of their appropriate triads, as discussed in § 1.

In order to discuss the stability of planetary waves completely the eigenvalue problem for (2.15) must be solved in the same manner as in § 4, extended to sufficiently large wavenumbers. The number of equations in each particular eigenvalue problem may be reduced by use of the selection rules, which show that interacting disturbance components may be grouped as shown in figure 2, and each of these groups treated separately. The eigenvalue problem for the onset of instability has been solved for all cases for all modes with $n = 3$ or 4 and for

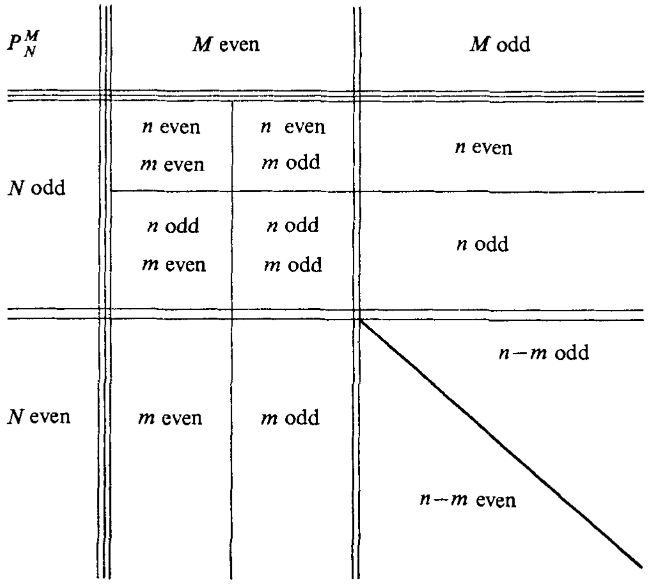

FIGURE 2. For a given primary wave P_N^M the boxes show independent groups of interacting disturbance waves P_n^m.

P_N^M	$N_T = 30$			Triad
P_3^1	0·789	n even	2·0	P_2^1 + zonal modes
P_3^2	0·403	n even, m odd	0·454	P_2^1, P_4^1
P_3^3	0·204	n even	0·204	P_2^1, P_4^2
P_4^1	0·082	$n-m$ even	0·773	P_3^1 + zonal modes
P_4^2	0·077	m odd	0·069	P_3^1, P_6^1
P_4^3	0·216	$n-m$ even	0·210	P_3^1, P_6^2
P_4^4	0·296	m even	0·458	P_3^2, P_6^2
P_5^4	0·040	n odd, m odd	0·040	P_3^1, P_7^3

TABLE 4. Critical amplitudes for instability of planetary waves obtained for $N_T = 30$, compared with the values from tables 2 and 3 for the most unstable triads.

the mode P_5^4, with the truncation number $N_T = 30$ for each case.† For this value of N_T the critical amplitudes seemed to be close to asymptotic values for large N_T. The critical amplitudes A_c obtained for the vorticity $A = 2\zeta_\alpha/\Omega$ are given in table 4, where they are compared with the corresponding triad values from tables 2 and 3. For each mode, these two figures are generally of the same order of magnitude and in most cases bear a striking similarity; for truncation numbers N_T less than 30 the similarity with the triad values is not as good (for the two waves where the comparison is least satisfactory, namely P_3^1 and P_4^1, the triad

† This ensures the inclusion of most modes for which $n \leqslant 8$, and for P_3^2 and P_5^4, all modes for which $n \leqslant 10$.

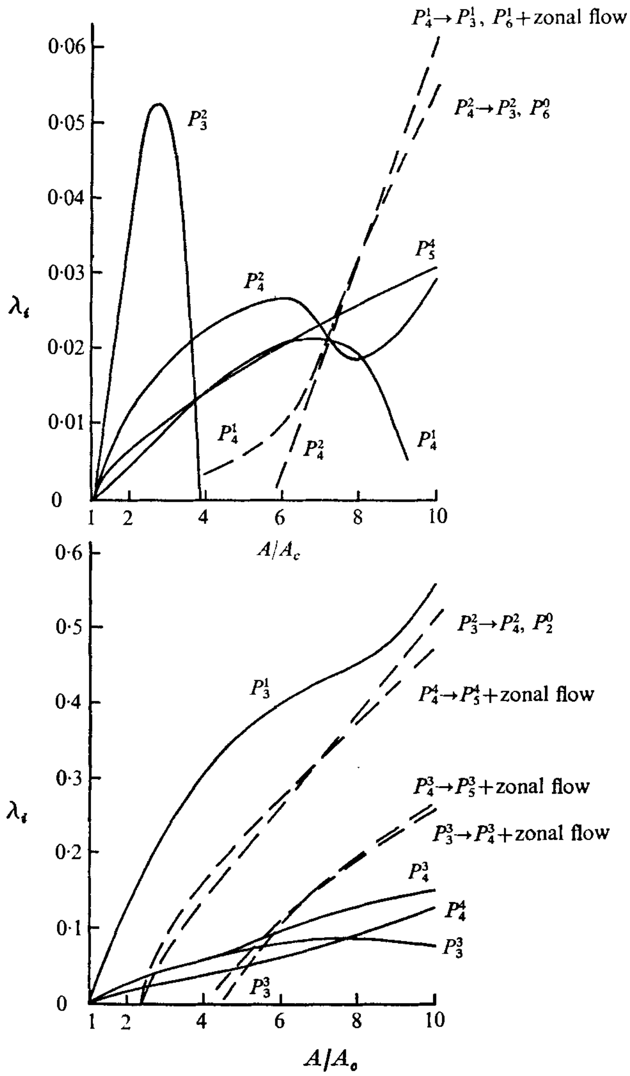

FIGURE 3. Growth rates λ_i (the growth factor for 1 day is exp $(2\pi\lambda_i)$) for the fastest-growing eigenfunctions for the unstable modes indicated, as a function of their vorticity amplitudes A scaled with their amplitudes A_c at the onset of instability, given in table 4. Continuous lines denote those eigenfunctions which become unstable when $A = A_c$, and for these eigenfunctions the components of the most unstable triad are prominent; dashed lines denote eigenfunctions with the highest growth rate for each primary wave when these are different from the eigenfunction which first becomes supercritical; some principal components of these latter eigenfunctions are indicated. Other unstable eigenfunctions appearing at supercritical values of A and with lower growth rates have been omitted.

is not a triad at all but consists of a planetary wave (P_2^1 or P_3^1) and all the interacting zonal modes; if the most unstable 'pure' triad is considered the critical amplitudes are closer to the $N_T = 30$ values). This similarity leads to the conjecture that the critical amplitudes for the most unstable triads for *all* planetary waves (excluding zonal flows) provide at least a good approximation to their critical amplitudes for the complete infinite system. The reasonableness of this supposition is supported by the experimental results of McEwan (1971) for the analogous system of a discrete spectrum of interacting internal waves in a finite tank: the critical amplitudes for forced internal waves agreed very well with their theoretical values obtained from calculations based on triads, after allowing for dissipation.

For every unstable planetary wave considered, the eigenfunction corresponding to the unstable eigenvalue at the onset of instability ($A = A_c$) contains the components of the most unstable triad as principal components. Growth rates for amplitudes larger than critical are shown in figure 3, where continuous lines refer to modes which commence growth at $A = A_c$. Other unstable eigenfunctions with different principal components may appear for supercritical values of A, and where these have growth rates greater than those of the initial eigenfunctions for $1 < A/A_c < 10$ they have been denoted by dashed lines. It is interesting to note that, as with the triad interactions, all the components of each unstable eigenfunction have the same time dependence (real and imaginary) when viewed relative to the frame rotating with the primary wave.

6. Numerical integrations for P_3^0 and P_5^4

Two integrations using the non-divergent form of the barotropic spectral model developed by Bourke (1972) were carried out to examine the behaviour of the system beyond the limits of linear theory, the first, for P_3^0, lasting for 12 days with a rhomboidal truncation number $J = 10$ (i.e. all waves with

$$|M| < J, \quad 1 < n < |M|+J$$

included) and a $\frac{1}{2}$ h time step, and the second, for P_5^4, lasting for 20 days with $J = 15$ and a 1 h time step. These truncation numbers were chosen on the basis of avoiding truncation effects as discussed by Puri & Bourke (1974), and economical computer time.

We consider first P_3^0, with positive vorticity coefficient (the atmospheric case), but before discussing the spectral-model integration we consider the behaviour of the 'most unstable triad' in isolation for both sub- and supercritical amplitudes. P_3^0 is the unstable member of a resonant triad containing waves P_2^2 and P_4^2, with a critical amplitude from linear theory (for the vorticity) of $1{\cdot}08\ \Omega$. Results of considering this triad in isolation are shown in figure 4 for two sets of initial amplitudes. If one writes

$$\zeta_3^0 = C, \quad \zeta_2^2 = \tfrac{1}{2}A\,e^{i\alpha}, \quad \zeta_4^2 = \tfrac{1}{2}B\,e^{i\beta}, \quad \theta = \beta - \alpha, \tag{6.1}$$

so that A, B and C denote the vorticity amplitudes† of the respective waves, the governing equations are

$$A = 0{\cdot}1543\,BC\sin\theta, \quad B = 0{\cdot}38576\,AC\sin\theta, \quad C = 0{\cdot}27003\,AB\sin\theta,$$

$$\theta = -0{\cdot}46667\,\Omega - 0{\cdot}5588\,C + (0{\cdot}38576\,A/B + 0{\cdot}1543\,B/A)\,C\cos\theta. \tag{6.2}$$

† The amplitude of the vorticity or stream function refers to $|\zeta_n^0|$ or $|\psi_n^0|$ for $m = 0$, and $2|\zeta_n^m|$ or $2|\psi_n^m|$ for $m \neq 0$.

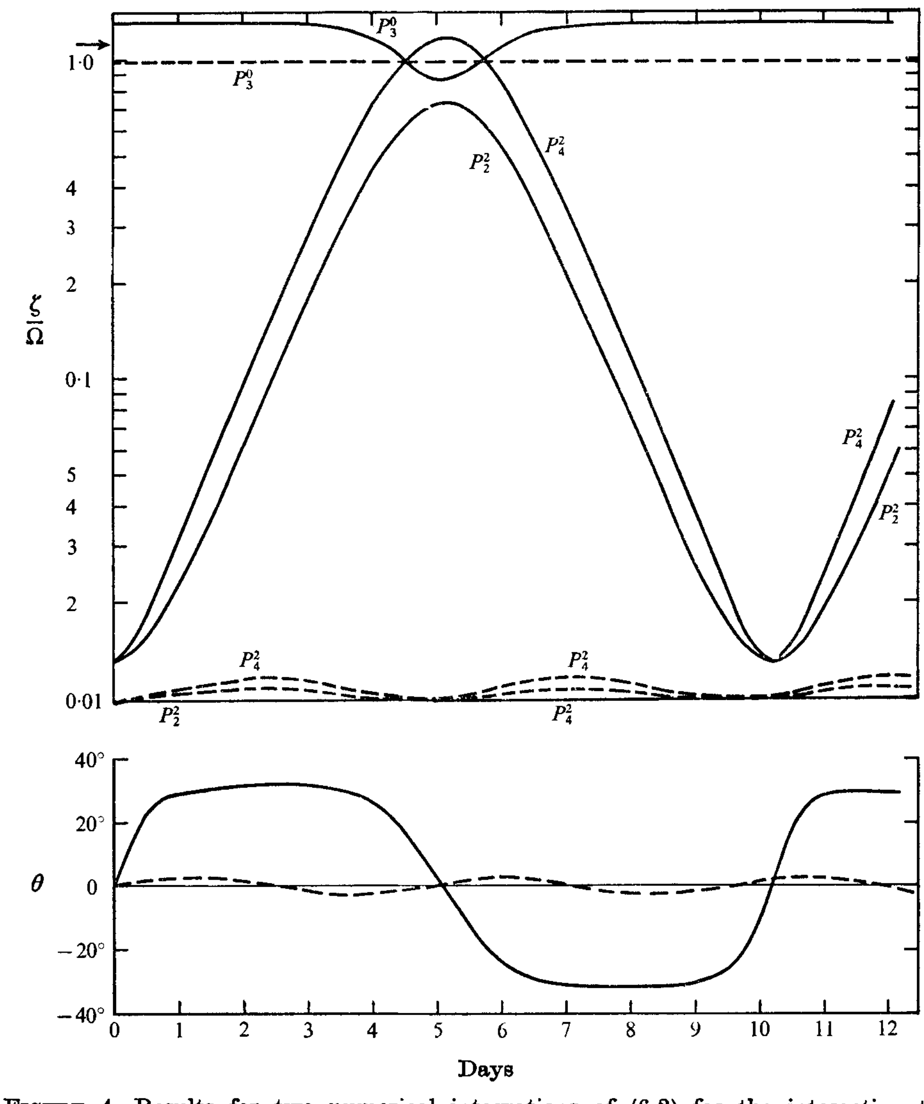

FIGURE 4. Results for two numerical integrations of (6.2) for the interacting triad (P_3^0, P_2^2, P_4^2) with the initial amplitudes of P_3^0 on either side of the critical amplitude (indicated by arrow) given by linear stability theory. The lowest curve shows the phase difference θ between ζ_4^2 and ζ_2^2.

These equations were integrated using a fourth-order Runge–Kutta method with a time step equivalent to 1 h for two amplitudes of P_3^0 on either side of the critical amplitude (0·983 Ω and 1·3 Ω) and with the amplitudes of P_2^2 and P_4^2 smaller by a factor of 100 in each case. The difference between the stable and unstable cases is evident: for the former, all the functions oscillate with only slight variations in amplitude, whereas for the latter the disturbance waves increase by two orders of magnitude. The functions are periodic with the expected elliptic-function (Platzman 1962) appearance.

The spectral-model integration, whose results are shown in figure 5, was

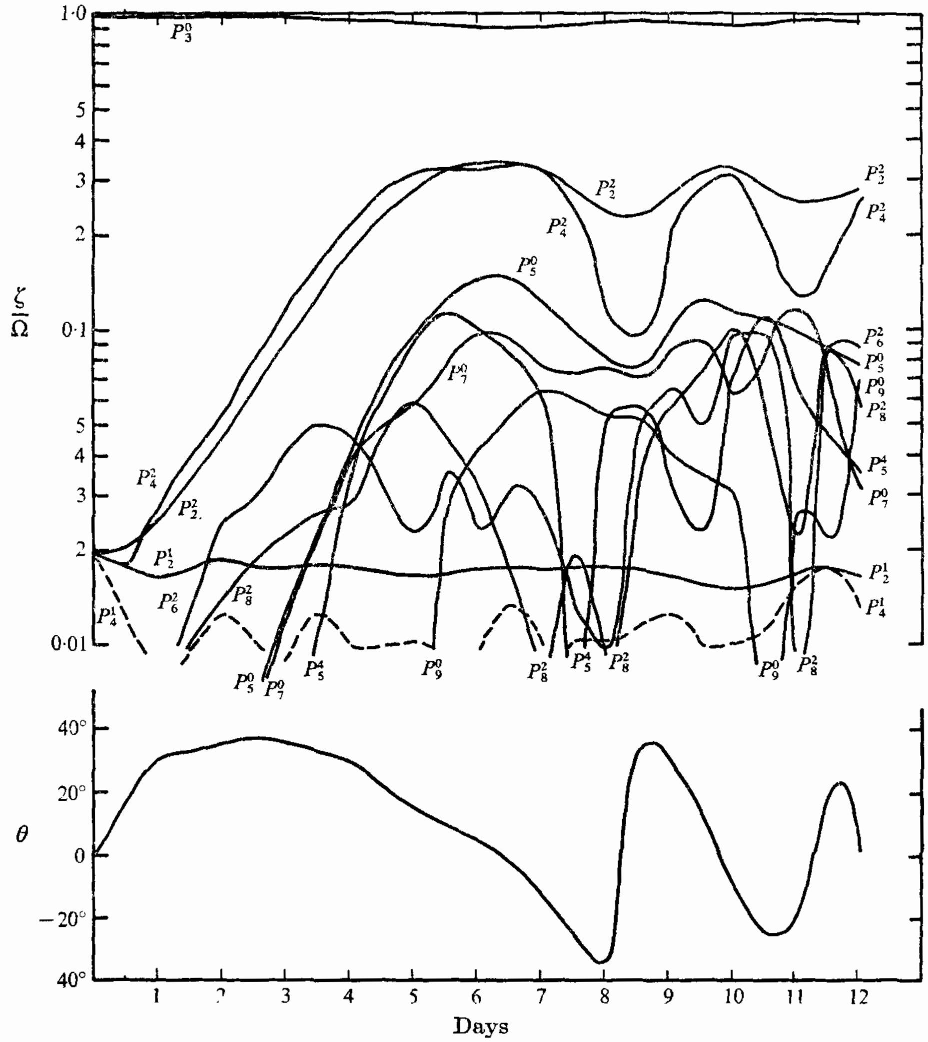

FIGURE 5. Results in terms of vorticity amplitudes from integration of a barotropic spectral model (see text) with primary wave $+P_3^0$ and disturbances as shown. The lowest curve shows the phase difference θ between ζ_4^2 and ζ_2^2, for comparison with figure 4.

begun with the same initial conditions as for the *stable* triad case, except that the waves P_2^1 and P_4^1 were added with the same initial amplitude and phase as P_2^2 and P_4^2. All waves which reach amplitudes above 0·01 Ω are shown, except for several whose amplitudes only just exceed this value. The critical amplitude for this degree of truncation ($N_T = 10$ in this case) is 0·805 Ω (*vs.* 0·799 Ω for $J > 20$), so that the P_3^0 mode is unstable; the modes P_2^2, P_4^2, P_2^2 and P_8^2 (and P_{10}^2 and P_{12}^2, which are not shown) all grow exponentially as components of the eigenvector comprising the destabilizing disturbance. The modes P_5^0, P_7^0 and P_9^0 together with P_5^4 appear at a later stage as a result of mutual interactions between the growing waves with $m = 2$ when their amplitudes have become appreciable. The waves P_2^1 and P_4^1 (which constitute a growing disturbance for ζ_3^0 negative)

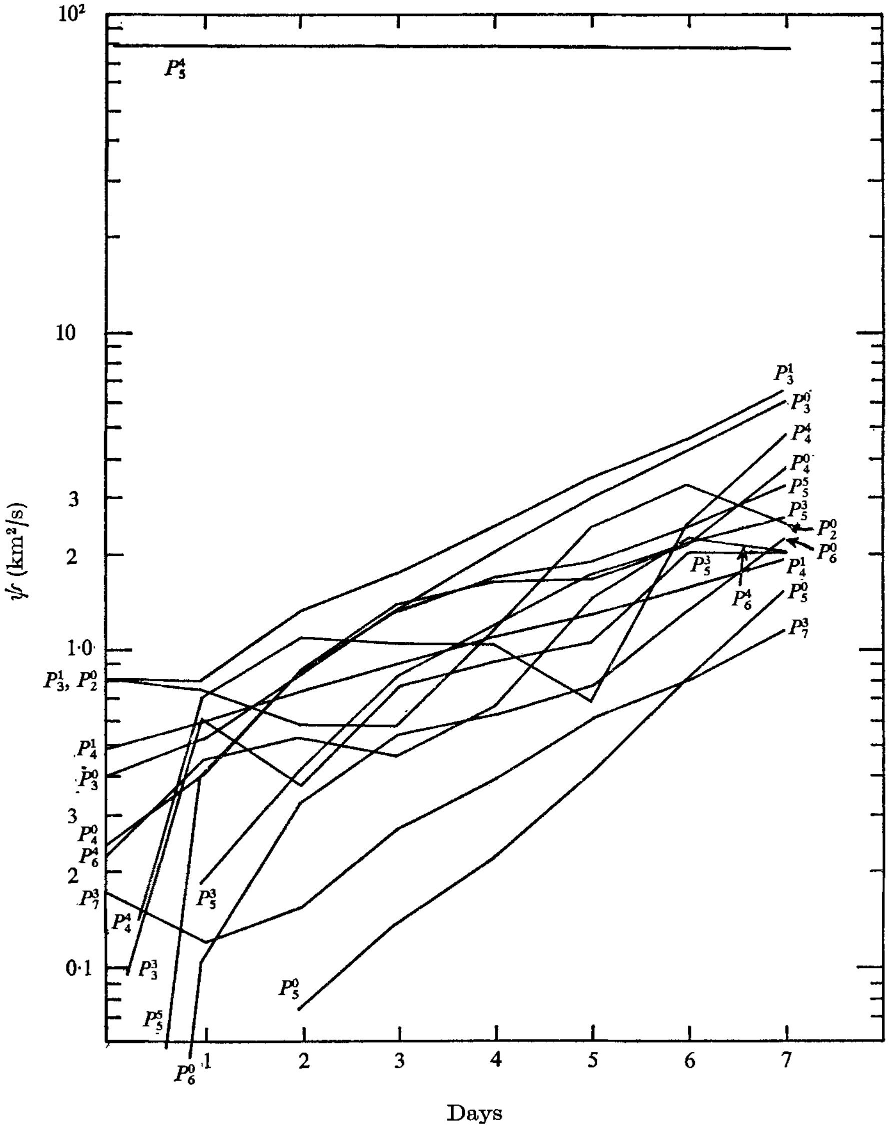

FIGURE 6. Results in terms of stream-function amplitudes with the earth's radius and rotation, from integration similar to that for figure 4 with primary wave P_5^4. With three exceptions, only modes with amplitudes greater than 2·0 at the end of the day 7 are shown, there being a multitude of curves at lower amplitudes exceeding 0·1.

initially oscillate like the stable triad of figure 3, and appear to interact only weakly with the other modes shown. In this integration the primary wave P_3^0 retains most of its initial energy (at least over 12 days) because, of the energy it loses, a substantial fraction must flow to waves with $n < 3$, i.e. $n = 2$. The only such mode interacting strongly with P_3^0 here is P_2^2, and this is clearly coupled to P_4^2, causing a quasi-oscillatory behaviour which limits its growth. In summary, as predicted by linear theory, the expansion of the spectrum has lowered the critical amplitude for P_3^0 below that predicted for the simple triad, but the growth of the disturbance has been limited to about one order of magnitude, apparently because of the small number of interacting modes with $n < 3$.

Results for the first 7 days of the integration with P_5^4 as the unstable wave are shown in figure 6, where the variable employed is the magnitude of the stream function in units of km²/s (related to the vorticity via the earth's radius). The initial amplitude of P_5^4 ($|\zeta/\Omega| = 0{\cdot}815$) is approximately the same as that used by Phillips (1959), and subsequently by several others, to test the efficacy of numerical models. The waves P_2^0, P_3^0, P_4^0, P_3^1, P_7^3, P_4^1 and P_6^4 were also introduced, all with vorticity coefficients ζ_n^m of the same magnitude and phase but much smaller than ζ_5^4 and 180° out of phase with it, yielding the initial stream-function amplitudes shown in figure 6.

At this amplitude ($A \sim 20\,A_c$) P_5^4 has a number of unstable eigenfunctions, the most rapidly growing of which is that which first becomes unstable when $A = A_c$ and has as its principal components P_3^1, P_3^3, P_7^3, P_7^5 and P_7^7. The growth rate for this mode is $\lambda_i = 0{\cdot}0803$ (the growth factor for 1 day is $\exp(2\Pi\lambda_i)$), yielding an e-folding growth time of 2 days, which is in good agreement with the observed growth rates of P_3^1 and P_7^3. Another unstable eigenfunction, with principal components P_4^0, P_4^4, P_6^0 and P_6^4, has a growth rate of 0·0418, giving an e-folding time of 3·8 days, so that the agreement for these modes is less satisfactory. Continuing the integration beyond day 7 shows that the amplitude of P_5^4 continues to decrease monotonically, so that half its energy has been lost by day 15 and it has completely lost its dominance by day 20, when several other modes have greater energies, and there is no indication that it would re-emerge from the large number of other modes with comparable amplitude.

Hence for P_5^4 at the above amplitude the growth rate of small disturbances as predicted by the linear theory is of a magnitude comparable with that obtained with a $J = 10$ spectral model. This growth rate is maintained beyond the limits of the linear theory, and for disturbances introduced at approximately 1 % of the amplitude of P_5^4, the 'destruction time' for the primary wave is of the order of 2–3 weeks.

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Appendix. The infinite β-plane

Gill (1974) has considered the stability of plane non-divergent Rossby waves on an infinite β-plane for small and large values of $M = UK^2/\beta$, where U is the velocity amplitude of the planetary wave and K is its wavenumber. He found them to be unstable in both these limits and, presumably, in between as well. For small M the appropriate form of disturbance consists of the two components of a resonantly interacting triad. This problem may be approached in a slightly different manner as follows.

Following Gill's notation, if the initial or primary wave is given by the stream function

$$\psi = (U/K)\sin\theta, \quad \theta = kx+ly+\omega t, \tag{A 1}$$

where

$$K^2\omega = \beta k, \quad K^2 = k^2+l^2, \tag{A 2}$$

the equation for the disturbance stream function ψ is

$$\nabla^2\psi_t+\beta\psi_x+(U/K)\cos\theta[k(\nabla^2\psi+k^2\psi)_y-l(\nabla^2\psi+k^2\psi)_x] = 0. \tag{A 3}$$

If we take K^{-1} and K/β as the units of length and time respectively the non-dimensional form of this equation is

$$\nabla^2\psi_t+\psi_x+M\cos\theta[\hat{k}(\nabla^2\psi+\psi)_y-\hat{l}(\nabla^2\psi+\psi)_x] = 0, \tag{A 4}$$

where $(\hat{k},\hat{l}) = (k/K, l/K)$. The most simple general solutions to this equation have the form

$$\psi = f(\theta)\,e^{i(px+qy+\eta t)}. \tag{A 5}$$

Substituting this in (A 4) gives the equation for f:

$$f'''+i(a_2+Mb_2\cos\theta)f''-(a_1+Mb_1\cos\theta)f'-i(a_0+Mb_0\cos\theta)f = 0, \tag{A 6}$$

where the coefficients a_i and b_i are functions of $\hat{k}$, $\hat{l}$, p, q and η. Equation (A 6) is a third-order version of the Mathieu equation, to which Floquet's theorem applies, so that it has solutions of the form $e^{im\theta}P(\theta)$, where $P(\theta)$ is periodic with period 2π. However, the exponential factor $e^{im\theta}$ may be regarded as being absorbed into the factor $e^{i(px+qy+\eta t)}$ so that it is sufficient to look for periodic solutions of (A 6). If M is small one may expand the solution in a Fourier series, substitute into (A 6) and equate coefficients of powers of M in a manner similar to (for example) Rhines (1970, § 2). The details are comparatively straightforward, and it transpires that if M is very small the equation has solutions with real p, q and η except when

$$a_0+na_1+n^2a_2+n^3 = O(M),$$

where n is an integer. If $n = 1$ it may be shown that this implies that two components of the Fourier series are $O(1)$ whilst the remainder are $O(M)$ and that these two components form a resonant triad with the primary Rossby wave. If p and q are regarded as real, η is complex; the growth rate (the imaginary part of η) increases linearly with M as M increases, as does the range of values of $(p^2+q^2)^{\frac{1}{2}}$ for which unstable solutions exist. Higher values of n correspond to higher-order wave interactions. The picture obtained is very similar to that of the classical Mathieu equation, and by analogy one would expect the unstable regions to broaden so as to cover almost all p and q as M increases.

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